Optimum Approximation for Residual Stiffness in Linear System Identification

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A new method of system identification uses projector matrix theory and the Moore-Penrose generalized inverse to derive an analytical stiffness matrix that, when combined with the analytical mass matrix, will more closely predict modal test results. Weighting matrices are used to enforce connectivity and make weighted corrections to the original analytical stiffness matrix. A simple and straightforward mathematical formulation is obtained. The method is compared and contrasted with other methods found in the literature, and a simple numerical example is presented.

Introduction

TODAY'S large and complex spacecraft require accurate analytical models for predicting dynamic loads. Finite element models may be generated with a high degree of precision; however, they lack accuracy due to uncertainty in material properties, joint properties, etc. The dynamic validity of the finite element model is determined by comparing the analytical frequencies and mode shapes with the results of a modal survey. If the correlation of the test and analysis modal parameters is unsatisfactory, the finite element model must be corrected.

Many systematic methods[1-13] have been developed in recent years for correcting analytical models to predict test results more closely. An approach that has met with some acceptance is the direct identification of the mass and stiffness matrices associated with the analytical model using test mode shapes and frequencies. Baruch and Bar Itzhack[4] took such an approach and developed a closed-form solution for the corrected stiffness matrix based on constrained minimization theory. Berman, Wei, and Rao[5] extended the constrained minimization technique to the correction of both the mass and stiffness matrices.

Both the Baruch and Berman methods offer several advantages such as a straightforward mathematical formulation and small required computational effort. Baruch's method produces a corrected stiffness matrix that, when combined with the corrected mass matrix generated by Berman's method, will exactly predict the test modes and frequencies used in the identification process. However, the simplicity of the method is overshadowed by several disadvantages in the correction of the stiffness matrix. Stiffness terms are changed in a nonrelative manner that can lead to undesirably large corrections in small terms. Also, the connectivity of the original analytical model is not preserved, causing the addition of unwanted load paths. These undesirable effects can lead to physically unrealistic changes in the analytical stiffness matrix.

Kabe[11] corrected the deficiencies of the Baruch method by modifying the assumed error function so that stiffness terms are corrected in a relative manner and the connectivity of the analytical model is preserved. Kabe[12] then expanded the method to include a weighting matrix expressing the confidence in the accuracy of each term in the stiffness matrix. The addition of model connectivity information to the constrained minimization formulation enables Kabe's method to identify the stiffness matrix exactly in certain cases even when some of the test modes are not known. This is possible using Baruch's method only if all the test modes are used in the identification process.

Kabe's improvements to the constrained minimization approach result in much more physically realistic corrections to the original analytical stiffness matrix. However, this method also possesses certain disadvantages. The mathematical formulation required to incorporate the discussed improvements, while being mathematically elegant, is very complicated. In addition, once the system identification problem is formulated, its solution requires a large amount of computational effort.

The purpose of the work presented in this paper was to develop a method of system identification having all the advantages of the Kabe method but possessing a much simpler mathematical formulation requiring less matrix manipulation prior to solution and less overall computational effort. The method discussed in this paper uses projector matrix theory[14] and the Moore-Penrose generalized inverse[14-16] to correct the analytical stiffness matrix. The method will be referred to as the Projector Matrix (PM) method. It is assumed that the analytical mass matrix, test modes, and test frequencies are accurate or have been adjusted using the techniques presented in the literature.[3-5]

In general, the PM method will yield a least-squares solution that minimizes an ellipsoidal norm of the difference between the analytical and corrected stiffness matrices. The minimization of a second ellipsoidal norm preserves analytical model connectivity. The method is consistent with constrained minimization theory but does not resort to the use of Lagrange multipliers as do the Baruch, Berman, and Kabe methods and the general least-squares formulation of Caesar.[13]

A simple numerical example is presented to compare the results obtained using the PM method and the methods developed by Baruch and Kabe. More difficult and physically representative system identification problems are presently being analyzed for future presentation. Because the Baruch and Berman methods use the same formulation for the correction of the stiffness matrix, the methods will be collectively referred to as the Baruch/Berman method throughout the remainder of this paper.

Received Nov. 20, 1986; presented as Paper 87-0813 at the AIAA/ASME/AHS/ASEE 28th Structures, Structural Dynamics and Materials Conference, Monterey, CA, April 6–8, 1987; revision received May 27, 1987.

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Theoretical Formulation

An undamped n-degree of freedom system is governed by the equation of motion

$$-M_n\Phi_n\omega_n^2+K_n\Phi_n=0 \qquad (1)$$

in which M_n and K_n are the system mass and stiffness matrices, respectively, ω_n^2 is a diagonal matrix containing system eigenvalues, and Φ_n is the modal matrix. If the mode shapes are normalized with respect to the mass matrix such that $\Phi_n^TM_n\Phi_n=I_n$ and if all n-mode shapes are contained in the modal matrix, the following relationship is valid

$$\Phi_n\Phi_n^TM_n=I_n \qquad (2)$$

Assuming that Φ_n are the test mode shapes, Φ_n can be partitioned into two sets, Φ_k and Φ_r. The set Φ_k contains all the modes that were measured during the modal survey. Set Φ_r contains the residual test modes that are unknown. This mode set partition allows Eq. (2) to be written in the form

$$U_k+U_r=I_n \qquad (3)$$

where

$$U_k=\Phi_k\Phi_k^TM_n \quad \text{and} \quad U_r=\Phi_r\Phi_r^TM_n$$

The same partitioning can be applied to the stiffness matrix K_n and the eigenvalue matrix ω_n^2 such that

$$K_n=K_k+K_r \qquad (4)$$

in which

$$K_k=M_n\Phi_k\omega_k^2\Phi_k^TM_n \qquad (5)$$

$$K_r=M_n\Phi_r\omega_r^2\Phi_r^TM_n \qquad (6)$$

Matrix K_k represents the contribution to the total stiffness due to the known test modes, while matrix K_r represents the residual portion of the stiffness matrix. If the test modes and frequencies and the analytical mass matrix are assumed to be correct and consistent, the stiffness matrix contribution K_k is known exactly. Therefore, only the residual stiffness due to the unknown test modes must be determined in order to identify the correct stiffness matrix.

The Baruch/Berman method[4-5] of system identification uses the method of Lagrange multipliers to obtain a corrected stiffness matrix K_B given by

$$K_B=K_a-K_a\Phi_k\Phi_k^TM_n-M_n\Phi_k\Phi_k^TK_a+M_n\Phi_k\omega_k^2\Phi_k^TM_n$$
$$+M_n\Phi_k\Phi_k^TK_a\Phi_k\Phi_k^TM_n \qquad (7)$$

in which K_a is the original analytical stiffness matrix. Using Eqs. (3) through (6), Eq. (7) can be written as

$$K_B=K_k+U_r^TK_aU_r \qquad (8)$$

Thus, the Baruch/Berman approximation of the residual stiffness is given by

$$K_{Br}=U_r^TK_aU_r \qquad (9)$$

Matrix U_r can be computed from test results by forming U_k and subtracting it from an n-dimensional identity matrix. Equation (8) can be directly derived without resorting to the added complication of constrained minimization theory and Lagrange multipliers by substituting Eqs. (3) through (6) into the equation of motion (1).

At this point it is useful to introduce the notion of an idempotent matrix,[14] which is defined as any square matrix E that has the property $E^2=E$. Both U_k and U_r are idempotent matrices. Matrix U_k possesses k eigenvalues with value 1 corresponding to the k-known mode shapes that span its range space and r eigenvalues with value 0 corresponding to the r unknown mode shapes that span its null space. The matrix, therefore, acts as a projector such that, if R^k represents the k-dimensional space spanned by the known test modes Φ_k, $KU_k=K$ if, and only if, the column space of K is contained in R^k. Matrix U_k is said to be the projector on Φ_k along Φ_r that is symbolized by P_{Φ_k,Φ_r}. Just the reverse is true for the matrix U_r. Projectors U_k and U_r are unique and orthogonal for the orthogonal subspaces Φ_k and Φ_r.

The Baruch/Berman relation for the residual stiffness (9) represents the projection of the original stiffness matrix K_a on the space spanned by the residual test modes. This approximation of the residual stiffness is very simple and straightforward; however, when combined with the known stiffness component K_k, structural connectivity is not preserved and original analytical stiffness terms are not changed in a relative manner.

The PM method of system identification is based on the expression

$$K_k=K_cU_k \qquad (10)$$

where K_c is the unknown stiffness matrix that, when postmultiplied by U_k, yields the stiffness due to the known test modes. Matrix K_k is the projection of K_c on the space spanned by Φ_k. The solution of Eq. (10) for the unknown stiffness terms in K_c requires that both K_k and K_c be expressed as column vectors F_k and X_c, respectively. Symmetry in matrices K_k and K_c is satisfied by considering only upper triangular terms from each matrix in the construction of F_k and X_c. Terms from K_k and K_c are written row-by-row into the respective column vectors.

Matrix Eq. (10) is transformed into the standard form

$$F_k=HX_c \qquad (11)$$

where H is a square matrix of order $s=n(n+1)/2$ containing terms from U_k. For a simple three-degree-of-freedom problem, column vectors F_k and X_c and coefficient matrix H are given by

$$F_k=\{K_{k11}K_{k12}K_{k13}K_{k22}K_{k23}K_{k33}\}^T$$
$$X_c=\{K_{c11}K_{c12}K_{c13}K_{c22}K_{c23}K_{c33}\}^T \qquad (12a)$$

$$H=\begin{bmatrix} U_{k11} & U_{k21} & U_{k31} & 0 & 0 & 0 \\ U_{k12} & U_{k22} & U_{k32} & 0 & 0 & 0 \\ U_{k13} & U_{k23} & U_{k33} & 0 & 0 & 0 \\ 0 & U_{k12} & 0 & U_{k22} & U_{k32} & 0 \\ 0 & U_{k13} & 0 & U_{k23} & U_{k33} & 0 \\ 0 & 0 & U_{k13} & 0 & U_{k23} & U_{k33} \end{bmatrix} \qquad (12b)$$

Due to the symmetry and sparseness of the terms within H, the matrix can be easily generated and efficiently stored.

Although Eq. (11) is based on the known stiffness component K_k, the PM method essentially determines an approximation for the residual stiffness K_r that is consistent with the test data, the analytical model connectivity, and the relative minimization of changes in the analytical stiffness terms.

Method of Solution

In general, the coefficient matrix H is singular due to the incompleteness of the test data; Eq. (11), therefore, cannot be solved simply by computing the inverse of H. There are not enough independent equations to determine a unique solution. The rank of H depends upon the number of test modes used in the identification process. If k test modes are used to identify the stiffness matrix K_c, the rank of the projector matrix U_k is k. The rank of H can be determined in the general case by inspecting the form given in Eq. (12b). The H matrix can be conveniently divided into n-row partitions where the i-th partition contains $(n-i+1)$ rows. Each partition contains the last $(n-i+1)$ columns of U_k as nonzero terms and, therefore, has a row rank equal to k or $(n-i+1)$, whichever is smaller for that particular partition. If this rank is added over all partitions, the rank of H in the general case is given by

$$R(H)=nk-k(k-1)/2 \tag{13}$$

The ratio of unknowns to equations can be reduced by eliminating known terms from the column vector X_c. In most cases, known terms correspond to zeros in K_c derived from the connectivity of the original analytical stiffness matrix K_a. Equation (11) can thus be partitioned into the form

$$F_k=[H' \mid H_0]\{X_c' \mid 0\}^T \tag{14}$$

The matrix equation that must be solved is now given by

$$F_k=H'X_c' \tag{15}$$

where X_c' contains m unknown stiffness terms and H' possesses m columns. If some nonzero stiffness terms are known, or assumed to be known, the right side of Eq. (15) can also be partitioned. The resulting known column vector is nonzero and must be subtracted from the left side of the equation to obtain the standard form.

Equation (15) will possess an exact solution and is said to be consistent if, and only if, X_c' lies in the column space of matrix H'. This will always be the case if the mode shapes used in the identification procedure are consistent with one another. The general solution is of the form

$$X_c'=X_{co}'+z \tag{16}$$

where X_{co}' represents the particular solution of Eq. (15) and z is the homogeneous solution that lies in the null space of H'. If H' is full-column rank, its null space contains only the zero vector, and the solution can be determined uniquely. Otherwise, an infinite number of solutions exist.

While analytical mode shapes are always consistent, test data contain errors in measurement and processing that can result in an inconsistent matrix equation. Therefore, an approximate solution to Eq. (15) is sought by computing a generalized inverse of the matrix H'. Penrose[15] showed that for every matrix A there is a unique matrix Y that satisfies the four equations

$$AYA=A \tag{17}$$

$$YAY=Y \tag{18}$$

$$(AY)^T=AY \tag{19}$$

$$(YA)^T=YA \tag{20}$$

The matrix Y is called the Moore-Penrose generalized inverse of A and is symbolized by $A^\dagger$. While there is only one Moore-Penrose inverse for any matrix A, an infinite number of generalized inverses of A exist that satisfy some but not all of Eqs. (17-20). These inverses will be represented using notation[14] of the form $A^{(i,j\ldots)}$ where i, j, etc. indicate which of the four equations are satisfied by the inverse. The Moore-Penrose inverse is therefore equivalent to $A^{(1,2,3,4)}$.

An optimum approximate solution to Eq. (15), termed the least-squares solution, is used in the PM method. A least-squares solution will minimize the Euclidean norm of the error vector

$$e=\|F_k-H'X_c'\| \tag{21}$$

The general least-squares solution of Eq. (15) by Theorem 1., page 104, of Ref. 14 is of the form

$$X_{LS}=H'^{(1,3)}F_k+[I-H'^{(1,3)}H']y \tag{22}$$

where $(I-H'^{(1,3)}H')=P_{N(H')}$ is a projector on the null space of H' and y is an arbitrary vector. If H' is not full column rank, there will be an infinite number of least-squares solutions. The optimum least-squares solution for the purpose of identifying a corrected stiffness matrix will minimize the Euclidean norm of the difference between terms in the analytical and corrected stiffness matrices given by

$$\Delta=\|X_a'-X_c'\| \tag{23}$$

where X_a' is a column vector containing all the nonzero upper triangular terms of the analytical stiffness matrix K_a.

The desired minimum norm least-squares solution can be derived by premultiplying Eq. (22) by H', yielding

$$H'X_{LS}=H'H'^{(1,3)}F_k=v \tag{24}$$

where it is obvious that X_{LS} satisfies the equation exactly, and v is contained in the range of H'. Reference 14, page 116, gives the unique solution to Eq. (24) that minimizes the norm of Eq. (23) as

$$X_{MN}=H'^{(1,4)}v+[I-H'^{(1,4)}H']X_a' \tag{25}$$

in which $[I-H'^{(1,4)}H']=P_{N(H')}$. Substituting for v from Eq. (24) gives

$$X_{MN}=H'^{(1,4)}H'H'^{(1,3)}F_k+[I-H'^{(1,4)}H']X_a' \tag{26}$$

which by Theorem 2., page 114, of Ref. 14 reduces to

$$X_{MN}=H'^{\dagger}F_k+[I-H'^{(1,4)}H']X_a' \tag{27}$$

The Moore-Penrose inverse $H'^{\dagger}$ is a (1,2,3,4) generalized inverse of H' that is also a (1,4) inverse. The relation for X_{MN} can therefore be written as

$$X_{MN}=H'^{\dagger}F_k+[I-H'^{\dagger}H']X_a' \tag{28}$$

If Eq. (15) is inconsistent, the solution given by Eq. (28) will not exactly preserve the model connectivity due to its least-squares nature. The enforcement of connectivity can be emphasized by introducing a diagonal positive definite weighting matrix W_s of order s and minimizing the ellipsoidal norm

$$\|F_k-H'X_c'\|_w^2=(F_k-H'X_c')^TW_s(F_k-H'X_c') \tag{29}$$

in the solution formulation rather than the Euclidean norm of Eq. (21). A premium is placed upon the satisfaction of equations involved in connectivity (i.e., equations corresponding to null terms in X_c by making the corresponding terms in W_s large relative to others.

As mentioned previously, it is desirable to minimize changes in the stiffness matrix in a relative sense. This can be generalized to the minimization of the ellipsoidal norm

$$\|X_a'-X_c'\|_u^2=(X_a'-X_c')^TU_m(X_a'-X_c') \tag{30}$$

in place of the Euclidean norm of Eq. (23), where U_m is a diagonal positive definite weighting matrix of order m. If U_{mii} is large relative to other weighting terms, a small change will be made in stiffness term X'_{ai}. A strictly relative minimization requires that each term U_{mii} is given by

$$U_{mii} = 1/(X'_{ai})^2$$

The ellipsoidal norms given by Eqs. (29) and (30) are incorporated into the solution by introducing the transformations[14]

$$H'' = W_s^{-1/2} H' U_m^{-1/2} \quad (31a)$$

$$X''_c = U_m^{1/2} X'_c \quad (31b)$$

$$F''_k = W_s^{1/2} F_k \quad (31c)$$

$$X''_a = U_m^{1/2} X'_a \quad (31d)$$

into the matrix equation $H'' X''_c = F''_k$ which is equivalent to Eq. (15). The solution formulation represented by Eqs. (24–28) can then be used to derive the desired soltuion for the optimally corrected stiffness matrix given by

$$X_{PM} = U_m^{-1/2} [W_s^{1/2} H' U_m^{-1/2}]^\dagger W_s^{1/2} F_k + U_m^{-1/2} [I - (W_s^{1/2} H' U_m^{-1/2})^\dagger (W_s^{1/2} H' U_m^{-1/2})] U_m^{1/2} X'_a \quad (32)$$

If the matrix equation is consistent, the weighting matrix W_s has no effect upon the solution. In the event that the coefficient matrix H' is full column rank, weighting matrix U_m does not affect the solution.

The Moore-Penrose inverse of H'' that appears in Eq. (32) is most efficiently computed using the Householder transformation.[17] If H'' is of rank t, it can be decomposed into the form $H'' = QV$ where Q is an $s \times t$ orthogonal matrix and V is upper triangular. The Moore-Penrose inverse is therefore given by

$$H''^\dagger = V^T [VV^T]^{-1} Q^T \quad (33)$$

This method is numerically stable and requires the inverse of only a relatively small $t \times t$ matrix. Also, the factored matrices Q and V can be stored efficiently.

Kabe's method[11-12] requires the inverse of a square singular matrix $[\alpha + \beta]$ of order $nk \times nk$. The inverse is computed using eigenvector decomposition resulting in a generalized inverse that is equivalent to a Moore-Penrose generalized inverse. The PM method requires the Moore-Penrose generalized inverse of a rectangular matrix H'' that by using the Householder transformation requires the inverse of a nonsingular matrix VV^T according to Eq. (33). The order of this matrix is equal to the rank of H given by Eq. (13) or the number of unknown stiffness terms m, whichever is smaller. The upper bound on the order of VV^T is therefore m. Kabe's matrix $[\alpha + \beta]$ has the same upper bound on its rank. If only one test mode is used in the correlation analysis, both matrices $[\alpha + \beta]$ and VV^T are the same size, $nk \times nk$. However, if more than one test mode is used, the matrix VV^T inverted in the PM method will be smaller than matrix $[\alpha + \beta]$, which must be inverted in Kabe's method. Matrix VV^T has a maximum size of $m \times m$ while matrix $[\alpha + \beta]$ grows as the number of test modes increases.

When the test data is consistent, Eq. (15) will also be consistent. If the matrix H' is full column rank, Eq. (32) will yield the unique exact solution. If H' is column rank deficient, the solution will exactly satisfy all the equations and minimize the norm in Eq. (30). If the test data is inconsistent and the rank of H is less than or equal to the number of unknowns m, Eq. (15) will still be consistent. The PM method will therefore still give a minimum norm solution that exactly predicts the test data, even though it is inconsistent. However, if the rank of H is greater than m, Eq. (15) will then be inconsistent and Eq. (32) will yield the unique least-squares solution. Solutions derived using Kabe's method possess these same characteristics.

As more inconsistent test modes are added to the correlation analysis, the best-fit nature of the least-squares solution used by the PM and Kabe methods tends to smooth out the inconsistencies between the test modes such as sign errors in small terms. The Baruch/Berman method, however, always enforces the exact prediction of the test data, whether it is consistent or not. This may not be a desirable feature if the test data possesses a large amount of error.

Additional comparison between the PM and Kabe methods indicates that matrix Eq. (10) used in the PM method is equivalent to the eigenvalue constraint imposed in Kabe's approach (Eq. 5 of Ref. 11). The ellipsoidal norm given by Eq. (30) is likewise equivalent to the error function in Eq. (3) of Ref. 12. It is therefore expected that the PM and Kabe methods are mathematically equivalent. The main difference between the methods is the mathematical formulation. It is believed by the author that the more simple and straightforward mathematical formulation used in the PM method results in a more efficient solution to the system stiffness identification problem.

The PM method is also similar to the generalized linear least-squares formulation introduced by Caesar.[13] Like the Baruch/Berman and Kabe methods, Caesar's approach also uses the method of Lagrange multipliers resulting in a formulation which is more complicated and less efficient than the PM method. Caesar, however, adjusts the number of unknown stiffness coefficients and the number of constraint equations such that the system does not possess more linearly independent equations than unknowns. In this manner, Caesar's method will always predict exactly the test data used in the correlation. The resulting corrected stiffness matrix will not necessarily be consistent with test data omitted from the correlation analysis. It seems more reasonable to enforce all the connectivity constraints and then compute a least-squares solution if the system is overdetermined and inconsistent, as do the PM and Kabe methods.

As a final point in the comparison of methods, the PM method has one other attribute not found in the other methods that have been mentioned. The addition of the weighting matrix W_s found in the ellipsoidal norm given by Eq. (29) allows the user to place a premium on the satisfac-

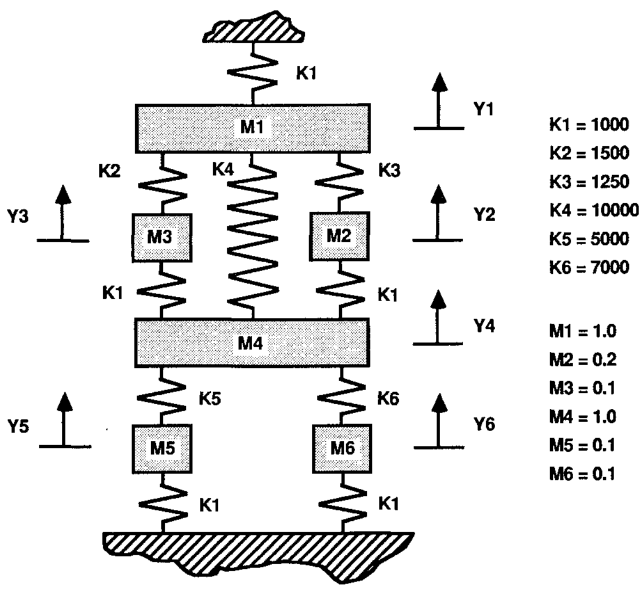

Fig. 1 Numerical example proposed by Kabe.[12]

tion of certain equations. In the case of a least-squares solution of an inconsistent matrix equation, the connectivity is not satisfied exactly even though the associated stiffness terms were assumed to be null in the original formulation. Additional emphasis can be placed upon the satisfaction of connectivity by placing a large weighting upon equations in F_k corresponding to null terms in K_c.

In general, the stiffness matrix used in a dynamic model is generated by reducing the original detailed model stiffness matrix to a smaller number of degrees of freedom using a transformation. This will generally result in a fully populated stiffness matrix. However, as suggested by Kabe,[11] many of the stiffness terms will be relatively small numbers that can be set to zero without changing the dynamic properties of interest. The increase in the number of known stiffness terms will, in general, result in a better solution using both the PM and Kabe mehtods for a fixed number of test modes.

It is also of interest to note here that, for an unrestrained structure, analytically generated rigid body modes can be used in conjunction with elastic body test modes to increase the rank of the coefficient matrix H', resulting in a better identification of the corrected stiffness matrix.

Numerical Example

A simple numerical example is used to illustrate the use of the PM method of system identification. The example selected, illustrated in Fig. 1, was taken from Kabe[12] so that a direct comparison could be made between the Kabe, Baruch/Berman, and PM methods.

Initially, an eigenvalue solution was performed for the six-degree-of-freedom example using the mass and stiffness coefficients listed in Fig. 1. The analytically derived modal parameters were used to simulate consistent test data. Stiffness terms were then corrupted to simulate an inaccurate analytical model, and the derived modal properties were used to simulate analytical data. Terms were corrupted by as much as 30%. Exact and corrupted stiffness terms are listed in Table 1. For this example, the H matrix used in the PM method will be of order 21 corresponding to the 21 upper-triangular terms in the stiffness matrix. Eight of these terms are known to be zero due to model connectivity, resulting in $m = 13$ unknown stiffness coefficients to be determined.

The PM and Baruch/Berman methods were applied to the example problem using one, two, and three consistent test modes in the indentification analysis and compared with results predicted by Kabe's method.[12] Table 1 lists the upper-triangular, adjusted-stiffness coefficients predicted by each method using only the first test mode shape and frequency. For one test mode, the H matrix will have a rank of six, which implies that the matrix H' will not be full column rank. The PM method therefore gives an exact solution to Eq. (15), which minimizes the ellipsoidal norm of Eq. (30). The results predicted by the PM and Kabe methods are comparable. Note that the Baruch/Berman method adds unwanted load paths to the adjusted stiffness matrix while the PM and Kabe methods preserve exactly the original model connectivity.

Table 2 lists the adjusted stiffness matrices using the first two consistent test modes and frequencies. In this case, the rank of the H matrix is 11, so H' will still be column rank deficient, resulting in an infinite number of exact solutions. The PM method will again yield the exact solution which minimizes the norm of Eq. (30). Stiffness matrix adjustments predicted by both the PM and Kabe methods show a significant increase in accuracy in many areas of the structure. In fact, in areas connecting degrees of freedom 5 and 6, the test modes and connectivity data provide enough information to identify exactly the associated stiffness coefficients.

Finally, all three methods were compared while using consistent test modes 1, 2, and 3. Table 3 presents the adjusted stiffness coefficients predicted for this case. The H matrix now has a rank of 15; consequently, two linearly independent columns are discarded while enforcing connectivity in the formation of H'. The PM method will therefore identify the unique and exact stiffness matrix for this case due to the consistency of the test data. Matrix H' will be full column rank whenever three or more test modes are used in the identification analysis for this example. Kabe's method also exactly identifies the stiffness coefficients; however, the Baruch/Berman method requires all the test modes and fre-

Table 1 Adjusted stiffness coefficients predicted using consistent test mode 1

		Adjusted stiffness coefficients			
Coeff. location	Corrupt coeff.	Baruch/ Berman	Kabe	PM	Exact coeff.
1,1	15750.	15576.	15543.	15540.	13750.
1,2	−1300.	−1314.	−1299.	−1299.	−1250.
1,3	−1300.	−1303.	−1298.	−1298.	−1500.
1,4	−12000.	−11939.	−12001.	−11999.	−10000.
1,5	0.	−22.	0.	0.	0.
1,6	0.	−63.	0.	0.	0.
2,2	2150.	2152.	2160.	2159.	2250.
2,3	0.	2.	0.	0.	0.
2,4	−850.	−813.	−849.	−848.	−1000.
2,5	0.	−3.	0.	0.	0.
2,6	0.	−12.	0.	0.	0.
3,3	2150.	2151.	2163.	2161.	2500.
3,4	−850.	−829.	−849.	−848.	−1000.
3,5	0.	−1.	0.	0.	0.
3,6	0.	−5.	0.	0.	0.
4,4	22900.	23184.	23310.	23326.	24000.
4,5	−4200.	−4202.	−4208.	−4211.	−5000.
4,6	−5000.	−5041.	−5037.	−5056.	−7000.
5,5	5100.	5098.	5066.	5071.	6000.
5,6	0.	−6.	0.	0.	0.
6,6	5900.	5890.	5787.	5809.	8000.

Table 2 Adjusted stiffness coefficients predicted using consistent test modes 1 and 2

		Adjusted stiffness coefficients			
Coeff. location	Corrupt coeff.	Baruch/ Berman	Kabe	PM	Exact coeff.
1,1	15750.	15554.	15850.	15654.	13750.
1,2	−1300.	−1305.	−1325.	−1318.	−1250.
1,3	−1300.	−1297.	−1558.	−1564.	−1500.
1,4	−12000.	−11940.	−12011.	−11810.	−10000.
1,5	0.	−19.	0.	0.	0.
1,6	0.	−59.	0.	0.	0.
2,2	2150.	2250.	2253.	2252.	2250.
2,3	0.	−25.	0.	0.	0.
2,4	−850.	−880.	−926.	−932.	−1000.
2,5	0.	−17.	0.	0.	0.
2,6	0.	−28.	0.	0.	0.
3,3	2150.	2155.	2461.	2457.	2500.
3,4	−850.	−813.	−899.	−888.	−1000.
3,5	0.	1.	0.	0.	0.
3,6	0.	−3.	0.	0.	0.
4,4	22900.	23228.	25874.	25664.	24000.
4,5	−4200.	−4194.	−5000.	−5000.	−5000.
4,6	−5000.	−5031.	−7000.	−7000.	−7000.
5,5	5100.	5099.	6000.	6000.	6000.
5,6	0.	−4.	0.	0.	0.
6,6	5900.	5892.	8000.	8000.	8000.

Table 3 Adjusted stiffness coefficients predicted using consistent test modes 1, 2, and 3

		Adjusted stiffness coefficients			
Coeff. location	Corrupt coeff.	Baruch/ Berman	Kabe	PM	Exact coeff.
1,1	15750.	13295.	13750.	13750.	13750.
1,2	−1300.	−1223.	−1250.	−1250.	−1250.
1,3	−1300.	−1182.	−1500.	−1500.	−1500.
1,4	−12000.	−9662.	−10000.	−10000.	−10000.
1,5	0.	−86.	0.	0.	0.
1,6	0.	−187.	0.	0.	0.
2,2	2150.	2247.	2250.	2250.	2250.
2,3	0.	−27.	0.	0.	0.
2,4	−850.	−960.	−1000.	−1000.	−1000.
2,5	0.	−16.	0.	0.	0.
2,6	0.	−26.	0.	0.	0.
3,3	2150.	2237.	2500.	2500.	2500.
3,4	−850.	−900.	−1000.	−1000.	−1000.
3,5	0.	−63.	0.	0.	0.
3,6	0.	−71.	0.	0.	0.
4,4	22900.	20940.	24000.	24000.	24000.
4,5	−4200.	−4148.	−5000.	−5000.	−5000.
4,6	−5000.	−4927.	−7000.	−7000.	−7000.
5,5	5100.	5149.	6000.	6000.	6000.
5,6	0.	−50.	0.	0.	0.
6,6	5900.	5949.	8000.	8000.	8000.

quencies to predict the exact solution. Table 4 lists frequencies computed using the adjusted stiffness matrices predicted by the PM method for all three cases using consistent data. In each case, the frequencies predicted for the modes used in the identification are exact. Significant improvement is seen in the frequencies of the residual modes for the two- and three-mode cases.

As mentioned previously, real test data are never consistent. As a result, the measure of the value of a system identification method is how well it performs using inconsistent modal data. Inconsistent test data were simulated by corrupting the exact analytical mode shapes with noise. Terms in each exact mode shape were scaled by the factor $1+\epsilon$ where ϵ was a random number with uniform probability distribution between $\pm\beta$.[11-12] The randomized modes were then corrected so that they were orthogonal to the analytical mass matrix using the method presented in Refs. 3 and 4 assuming knowledge of only the first four modes.

Initially, β was assumed to be 0.05, simulating a 5% noise level. The adjusted stiffness coefficients predicted by the PM method using the first three randomized test modes are listed in Table 5. In this case, the rank of the H matrix is greater than the number of unknown stiffness coefficients as discussed previously during the identification analysis using consistent test data. However, now the test data are inconsistent, so the PM method will yield a least-squares solution. The adjusted stiffness coefficients predicted using test data containing 5% noise compare well with the exact values computed for noise-free test data.

A cross-orthogonality check was performed comparing the 5% noise level test mode shapes with the corrupted system modes, and the mode shapes computed using the corrected stiffness matrix. The results, listed in Table 6, indicate that test modes 1 and 2 are accurately predicted by the corrupted model. Test modes 3 and 4, however, possess cross-generalized mass values much less than 1.0 and large coupling terms, indicating that the corrupted analytical model does not predict these modes. The corrected model modes possess cross-generalized mass values of 1.0 (accuracy to three decimal places) and small coupling terms for all four test modes even though only the first three modes were used in the identification process. A comparison of frequencies for the corrupted and adjusted analytical models is presented in Table 7. Note that the modes used in the identification process are not exactly predicted by the adjusted model; however, the results presented in Tables 5 through 7 indicate that the PM method accurately predicts a corrected analytical model using test data containing a 5% noise level. These results are comparable to those predicted by Kabe.[12]

The impact of the test data noise level upon the PM method was further explored by increasing the value of β to 0.10 and 0.20, simulating test data with 10% and 20% noise levels. Adjusted stiffness coefficients for both cases using the first three test modes are also presented in Table 5. At the 10% noise level, the predicted stiffness corrections tend to be inaccurate in areas of the structure containing low modal strain energy in the test modes and high modal strain energy in the residual modes.

In this example, degree-of-freedom 5 possesses 86% of the strain energy in residual mode 5, and degree-of-freedom 6 possesses 82% of the strain energy in residual mode 6. Both of these degrees of freedom possess a small amount of modal strain energy for the test modes used in the identification process. The largest inaccuracies in the adjusted stiffness matrix for the 10% noise level appear to be related to degrees of freedom 5 and 6. Frequencies predicted using the adjusted analytical model are listed in Table 7. Frequencies associated with modes used in the identification are predicted with reasonable accuracy, while those associated with residual modes possess a larger error.

At the 20% noise level, the corrections to the analytical stiffness matrix predicted by the PM method become physically unrealistic, resulting in an indefinite system and a negative eigenvalue corresponding to a residual mode as

Table 4 Comparison of frequencies for adjusted models predicted by PM method

Corrupt model	Model Adjusted Using: Mode 1	Modes 1,2	Modes 1,2,3	Exact model
5.216	5.258	5.258	5.258	5.258
17.204	17.236	17.626	17.626	17.626
23.514	23.553	24.285	23.171	23.171
25.495	25.444	26.369	26.060	26.060
36.798	36.651	40.207	40.169	40.169
41.256	41.122	47.544	47.464	47.464

Table 5 Adjusted stiffness coefficients predicted by PM method using inconsistent test modes 1, 2, and 3

Coeff. location	Corrupt coeff.	Adjusted stiffness coefficients: 5%	10%	20%	Exact coeff.
1,1	15750.	13706.	14456.	12362.	13750.
1,2	−1300.	−1198.	−1197.	−1538.	−1250.
1,3	−1300.	−1448.	−1147.	−1357.	−1500.
1,4	−12000.	−10013.	−9824.	−9643.	−10000.
1,5	0.	0.	0.	0.	0.
1,6	0.	0.	0.	0.	0.
2,2	2150.	2266.	2242.	2213.	2250.
2,3	0.	0.	0.	0.	0.
2,4	−850.	−969.	−1108.	−892.	−1000.
2,5	0.	0.	0.	0.	0.
2,6	0.	0.	0.	0.	0.
3,3	2150.	2504.	2277.	2208.	2500.
3,4	−850.	−948.	−1055.	−1082.	−1000.
3,5	0.	0.	0.	0.	0.
3,6	0.	0.	0.	0.	0.
4,4	22900.	21924.	18040.	18375.	24000.
4,5	−4200.	−4807.	−2341.	−13097.	−5000.
4,6	−5000.	−5852	−4873.	−6615.	−7000.
5,5	5100.	5930.	3642.	14872.	6000.
5,6	0.	0.	0.	0.	0.
6,6	5900.	7279.	5706.	−4861.	8000.

Table 6 Comparison of cross-generalized mass (CGM) between inconsistent test modes and modes predicted by uncorrected models (5% noise level)

Test mode	Analysis mode	Corrupt System			Corrected System		
		CGM	Coupling	Mode	CGM	Coupling	Mode
1	1	1.000	−0.012	2	1.000	0.001	4
2	2	−1.000	0.013	4	1.000	−0.007	4
3	3	−0.707	0.704	4	1.000	−0.002	4
4	4	−0.707	−0.706	3	1.000	0.007	2

Table 7 Comparison of frequencies for adjusted models predicted by PM method using inconsistent test modes 1, 2, and 3

Corrupt model	Models adjusted using test data with noise			Exact model
	5%	10%	20%	
5.216	5.242	5.173	5.476	5.258
17.204	17.635	17.629	17.589	17.626
23.514	23.168	23.154	23.195	23.171
25.495	26.023	24.764	24.576	26.060
36.798	39.795	31.159	−37.486*	40.169
41.256	45.096	39.920	63.858	47.464

*Corresponds to negative system eigenvalue.

presented in Table 7. These results indicate that a 20% noise level in the test data must be considered excessive for use in the PM method of system identification for this example. These results are also comparable to those predicted by Kabe.[12]

Conclusion

A new method of linear system identification possessing a simple and straightforward mathematical formulation has been presented. The PM method uses projector matrix theory and the Moore-Penrose generalized inverse to derive a corrected stiffness matrix that, when combined with the analytical mass matrix, will predict modal test parameters more closely. Weighting matrices are used to enforce model connectivity and make weighted corrections to the original analytical stiffness matrix. If the modal test data used in the identification yields insufficient or just sufficient information to identify uniquely the unknown stiffness coefficients, the adjusted stiffness matrix predicted by the PM method will predict exactly the test modes and preserve connectivity exactly whether or not the test data are consistent. If the test data yield more than enough information for unique identification and if the test data are inconsistent, the PM method will result in a least-squares solution.

The PM method was successfully applied to a simple example problem and compared with results predicted by the Kabe and Baruch/Berman methods for consistent test data. Results predicted by the PM method were consistent with those predicted by the Kabe method, indicating that the methods are mathematically equivalent. Real test data were simulated at the 5%, 10%, and 20% noise levels for the first four modes. The adjusted stiffness matrix predicted by the PM method using the 5% noise level test data accurately predicted all the test modes and frequencies. The PM method of system identification, therefore, gives accurate results for test data which are reasonably accurate. As the noise level was increased to 10%, substantial errors appeared in the residual portion of the structure. At the 20% noise level, physically unrealistic stiffness corrections were predicted, indicating that significant errors were present in the test data.

The PM method possesses a simple mathematical formulation that allows the method to be very easily implemented using FORTRAN and available finite element codes. However, like the Kabe method, the PM method corrects the stiffness matrix on a term-by-term basis, which can result in large matrices for moderately sized identification problems. To date, the method has been used only on relatively small problems. As a result, no data are available indicating the amount of computer resources required to use the method on a realistic problem. Work is currently being performed in this area for future presentation.

References

[1]Berman, A. and Flannelly, W. G., "Theory of Incomplete Models of Dynamic Structures," *AIAA Journal*, Vol. 9, No. 8, Aug. 1971, pp. 1482–1487.

[2]Berman, A., "System Identification of a Complex Structure," AIAA Paper 75-809, Denver, CO, May 1975.

[3]Targoff, W. P., "Orthogonality Check and Correction of Measured Modes," *AIAA Journal*, Vol. 14, Feb. 1976, pp. 164–167.

[4]Baruch, M. and Bar Itzhack, I. Y., "Optimal Weighted Orthogonalization of Measured Modes," *AIAA Journal*, Vol. 16, April 1978, pp. 346–351.

[5]Berman, A., Wei, F., and Rao, K. V., "Improvement of Analytical Dynamic Models Using Modal Test Data," AIAA Structures, Structural Dynamics, and Materials Conference, Paper 80-0800, 1980, pp. 809–814.

[6]Wei, F. S., "Stiffness Matrix Correction from Incomplete Test Data," *AIAA Journal*, Vol. 18, No. 10, Oct. 1980, pp. 1274–1275.

[7]Baruch, M., "Optimal Correction of Mass and Stiffness Matrices Using Measured Modes," *AIAA Journal*, Vol. 20, No. 11, Nov. 1982, pp. 1623–1626.

[8]Baruch, M., "Methods of Reference Basis for Identification of Linear Dynamic Structures," AIAA Paper 82-0769, New Orleans, LA, May 1982.

[9]Berman, A. and Nagy, E. J., "Improvement of a Large Analytical Model Using Test Data," *AIAA Journal*, Vol. 21, No. 8, Aug. 1983, pp. 1168–1173.

[10]Berman, A., "System Identification of Structural Dynamic Models—Theoretical and Practical Bounds," AIAA/ASME/ASCE/AHS 25th Structures, Structural Dynamics and Materials Conference, Paper 84-0929, Palm Springs, CA, May 1984, pp. 123–129.

[11]Kabe, A., "Stiffness Matrix Adjustment Using Mode Data," *AIAA Journal*, Vol. 23, No. 9, Sept. 1985, pp. 1431–1436.

[12]Kabe, A., "Constrained Adjustment of Analytical Stiffness Matrices," SAE Paper 851932.
[13]Caesar, B., "Update and Identification of Dynamic Mathematical Models," 4th International Modal Analysis Conference, Los Angeles, CA, Feb. 1986, pp. 394–401.
[14]Greville, T. N. E. and Ben-Israel, A., *Generalized Inverses: Theory and Applications*, Wiley, New York, NY, 1974.
[15]Penrose, R., "A Generalized Inverse for Matrices," Proceedings of the Cambridge Philosophical Society, Vol. 51, 1955, pp. 406–413.
[16]Greville, T. N. E., "The Pseudoinverse of a Rectangular or Singular Matrix and Its Application to the Solution of Systems of Linear Equations," *SIAM Review*, Vol. 1, No. 1, Jan 1959, pp. 38–43.
[17]Noble, B., *Generalized Inverses and Applications*, Academic Press, New York, NY, 1976.